

Relativistic spinor equation of photon

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Abstract

In this paper, we have proposed the spiron equation of free and non-free photon, and give the spin operator and spin wave function of photon. We calculate the helicity of photon and prove there are left-handed and right-handed photon. By the spiron equation of non-free photon, we can study the quantum property of photon in medium, which can be used in quantum optics, photonic crystals and so on.

Keywords: Photon spiron equation; Spin operator; Spin wave function

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1 Introduction

Maxwell has unified the laws of electricity and magnetism in a consistent way into a set of four equations, and he calculated the speed of electromagnetic waves and found that it was $300,000\text{km/sec}$, which was the same of light. This forced Maxwell to ponder about the nature of light: he concluded that light is a form of electromagnetic wave. For the explanation the spectra of black-body radiation, Planck firstly proposed that emission of black-body was energy quantization with value of $\hbar\omega$, and he introduced the Planck constant \hbar [1]. In photoelectric effect experiment, Einstein only considered the energy conservation and made use of Planck's $\hbar\omega$ to propose a quanta concept of light [2], and successfully explaining photoelectric effect phenomena, It was the beginning of Quantum Physics.

Despite claims that massless particles cannot be localized in space-time [3, 4], physicists have pondered over the problem of photon localization and the related problem of the photon wave

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function for almost 80 years now, beginning with the work of Landau and Peierls [5]. An extensive review of the photon localization problem was presented by Keller [6]. The problem of photon localization is closely related to the widely studied problem of the photon position operator. Quantum optics has long struggled to define a quantum wavefunction for photons that would be in agreement with innumerable experiments revealing the possibility of photon localization. The photon wave function appears in both of these descriptions and that it provides a very convenient concept to unify the two points of view. A very useful mathematical tool in this analysis is the Riemann-Silberstein vector [7–11]. Applications of the RS vector to many physical problems were reviewed in a very thorough paper by Keller [6].

After Dirac discovered the relativistic equation for a particle with spin 1/2 [12], much work was done to study spinor and vectors within the Lorentz group theory. Any quantity which transforms linearly under Lorentz transformations is a spinor. For that reason spinor quantities are considered as fundamental in quantum field theory and basic equations for such quantities should be written in a spinor form. A spinor formulation of Maxwell equations was studied by Laporte and Uhlenbeck [13], also see Rumer [14]. In 1931, Oppenheimer [15] proposed to consider the Maxwell theory of electromagnetism as the wave mechanics of the photon. They introduced a complex 3-vector wave function satisfying the massless Dirac-like equations. In this paper, we shall give the relativistic spinor equation for photon with spin 1 and mass 0.

2 Relativistic spinor equation of free photon

Dirac derived the Dirac equation by factorizing Einstein's dispersion relation such that the field equation becomes the first order in time derivative [16]. Namely, he factorized the relativistic dispersion relation employing four by four matrices

$$E^2 - c^2 \vec{p}^2 - m_0^2 c^4 = (E - c\vec{p} \cdot \vec{\alpha} - m_0 c^2 \beta)(E + c\vec{p} \cdot \vec{\alpha} + m_0 c^2 \beta), \quad (1)$$

where $\vec{\alpha}$ and β are Dirac matrices. For a photon, mass $m_0 = 0$, equation (1) becomes

$$E^2 - c^2 \vec{p}^2 = (E - c\vec{\alpha} \cdot \vec{p})(E + c\vec{\alpha} \cdot \vec{p}), \quad (2)$$

i.e.,

$$E - c\vec{\alpha} \cdot \vec{p} = 0, \quad (3)$$

canonical quantization equation (3), we have

$$i\hbar \frac{\partial}{\partial t} \psi = -i\hbar c \vec{\alpha} \cdot \vec{\nabla} \psi = H\psi, \quad (4)$$

where the photon's Hamiltonian operator $H = -i\hbar c \vec{\alpha} \cdot \vec{\nabla}$, and ψ is spinor wavefunction of photon. For the proper Lorentz group L_p , the irreducibility representations of spin 1 photon is D^{10} , D^{01} and

$D^{\frac{1}{2}\frac{1}{2}}$, respectively, and the dimension of irreducibility representations are corresponding to three, three and four. We choose photon's spiron wave function as the basis vector of three dimension irreducibility representations, it is

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (5)$$

and $\vec{\alpha}$ matrices are taken as

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\hbar \\ 0 & i\hbar & 0 \end{pmatrix}, \alpha_y = \begin{pmatrix} 0 & 0 & i\hbar \\ 0 & 0 & 0 \\ -i\hbar & 0 & 0 \end{pmatrix}, \alpha_z = \begin{pmatrix} 0 & -i\hbar & 0 \\ i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (6)$$

they are Hermitian matrices

$$\vec{\alpha}^\dagger = \vec{\alpha}, \quad (7)$$

and photon's Hamiltonian operator is Hermitian

$$H^\dagger = H. \quad (8)$$

3 The spin operators of photon

In the following, we shall prove the selection of $\vec{\alpha}$ matrices are reasonable, they confirm that equation (4) is photon's spiron equation of spin 1. equation (4) can be written as

$$i\hbar \frac{\partial}{\partial t} \psi = c(\vec{p} \cdot \vec{\alpha}) = H\psi, \quad (9)$$

where

$$H = c\vec{p} \cdot \vec{\alpha}, \quad (10)$$

The orbital angular momentum of photon satisfy

$$\begin{aligned} \frac{d}{dt} L_x &= \frac{1}{i\hbar} [L_x, H] \\ &= c(\alpha_y p_z - \alpha_z p_y) \\ &= c(\vec{\alpha} \times \vec{p})_x, \end{aligned} \quad (11)$$

i.e.,

$$[\vec{L}, H] = i\hbar c(\vec{\alpha} \times \vec{p}), \quad (12)$$

the equation (12) is shown that the orbital angular momentum of photon isn't conservation, but the total angular momentum of free photon should be conservative. So, photon should have an intrinsic

angular momentum, i.e., spin angular momentum \vec{s} , the total angular momentum of photon \vec{J} is

$$\vec{J} = \vec{L} + \vec{s}, \quad (13)$$

and \vec{J} is conservative

$$[\vec{J}, H] = 0, \quad (14)$$

with equations (12) and (14), we have

$$[\vec{s}, H] = -[\vec{L}, H] = -i\hbar c(\vec{\alpha} \times \vec{p}), \quad (15)$$

i.e.,

$$\begin{aligned} [s_x, H] &= [s_x, c\vec{\alpha} \cdot \vec{p}] = -i\hbar c(\vec{\alpha} \times \vec{p})_x \\ &= i\hbar c(\alpha_z p_y - \alpha_y p_z), \end{aligned} \quad (16)$$

or

$$\begin{aligned} [s_x, c\alpha_x p_x + c\alpha_y p_y + c\alpha_z p_z] &= c[s_x, \alpha_x]p_x + c[s_x, \alpha_y]p_y + c[s_x, \alpha_z]p_z \\ &= i\hbar c(\alpha_z p_y - \alpha_y p_z), \end{aligned} \quad (17)$$

there are commutation relations

$$[s_x, \alpha_x] = 0, \quad [s_x, \alpha_y] = i\hbar\alpha_z, \quad [s_x, \alpha_z] = -i\hbar\alpha_y, \quad (18)$$

it is easy to summarize that

$$[s_i, \alpha_j] = i\hbar\epsilon_{ijk}\alpha_k, \quad (19)$$

with equation (18), we can calculate the s matrices, let the s_x matrix is

$$s_x = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad (20)$$

with the commutation relation

$$[s_x, \alpha_y] = i\hbar\alpha_z, \quad [s_x, \alpha_z] = -i\hbar\alpha_y, \quad (21)$$

we get

$$s_x = \begin{pmatrix} a & 0 & 0 \\ 0 & a & -i\hbar \\ 0 & i\hbar & a \end{pmatrix}, \quad (22)$$

let the s_y matrix is

$$s_y = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad (23)$$

with the commutation relation

$$[s_y, \alpha_x] = -i\hbar\alpha_z, \quad [s_y, \alpha_z] = i\hbar\alpha_x, \quad (24)$$

we get

$$s_y = \begin{pmatrix} b & 0 & i\hbar \\ 0 & b & 0 \\ -i\hbar & 0 & b \end{pmatrix}, \quad (25)$$

let the s_z matrix is

$$s_z = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \quad (26)$$

with the commutation relation

$$[s_z, \alpha_x] = i\hbar\alpha_y, \quad [s_z, \alpha_y] = -i\hbar\alpha_x, \quad (27)$$

we get

$$s_z = \begin{pmatrix} c & -i\hbar & 0 \\ i\hbar & c & 0 \\ 0 & 0 & c \end{pmatrix}. \quad (28)$$

In the following, we should calculate the eigenvalues of s_x, s_y, s_z .
The s_x eigenvalue problem $s_x\psi = \lambda_1\psi$ is

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & -i\hbar \\ 0 & i\hbar & a \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \lambda_1 \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (29)$$

therefore the characteristic equation is

$$\begin{vmatrix} a - \lambda_1 & 0 & 0 \\ 0 & a - \lambda_1 & -i\hbar \\ 0 & i\hbar & a - \lambda_1 \end{vmatrix} = 0, \quad (30)$$

i.e.,

$$(a - \lambda_1)[(a - \lambda_1)^2 - \hbar^2] = 0, \quad (31)$$

when $a = 0$, the roots λ_1 are

$$\lambda_1 = \pm \hbar, \quad (32)$$

the s_y eigenvalue problem $s_y \psi = \lambda_2 \psi$ is

$$\begin{pmatrix} b & 0 & i\hbar \\ 0 & b & 0 \\ -i\hbar & 0 & b \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \lambda_2 \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (33)$$

therefore the characteristic equation is

$$\begin{vmatrix} b - \lambda_2 & 0 & i\hbar \\ 0 & b - \lambda_2 & 0 \\ -i\hbar & 0 & b - \lambda_2 \end{vmatrix} = 0, \quad (34)$$

i.e.,

$$(b - \lambda_2)[(b - \lambda_2)^2 - \hbar^2] = 0, \quad (35)$$

when $b = 0$, the roots λ_2 are

$$\lambda_2 = \pm \hbar, \quad (36)$$

the s_z eigenvalue problem $s_z \psi = \lambda_3 \psi$ is

$$\begin{pmatrix} c & -i\hbar & 0 \\ i\hbar & c & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \lambda_3 \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (37)$$

therefore the characteristic equation is

$$\begin{vmatrix} c - \lambda_3 & -i\hbar & 0 \\ i\hbar & c - \lambda_3 & 0 \\ 0 & 0 & c - \lambda_3 \end{vmatrix} = 0, \quad (38)$$

i.e.,

$$(c - \lambda_3)[(c - \lambda_3)^2 - \hbar^2] = 0, \quad (39)$$

when $c = 0$, the roots λ_3 are

$$\lambda_3 = \pm \hbar. \quad (40)$$

By calculation, we obtain the spin matrices of photon, they are

$$s_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\hbar \\ 0 & i\hbar & 0 \end{pmatrix}, s_y = \begin{pmatrix} 0 & 0 & i\hbar \\ 0 & 0 & 0 \\ -i\hbar & 0 & 0 \end{pmatrix}, s_z = \begin{pmatrix} 0 & -i\hbar & 0 \\ i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (41)$$

Obviously, these matrices are photon's spin matrices, which describe the spin $s = 1$. On the one hand, their eigenvalues are $\pm\hbar$. On the other hand, the matrices square is

$$\vec{s}^2 = s_x^2 + s_y^2 + s_z^2 = 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \hbar^2 = s(s+1)\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (42)$$

i.e., the spin s is

$$s = 1. \quad (43)$$

Comparing (6) with (41), we find

$$s_x = \alpha_x, \quad s_y = \alpha_y, \quad s_z = \alpha_z. \quad (44)$$

4 The helicity of photon

By studying the helicity of photon, we can obtain the photon left-handed and right-handed. The helicity is defined as the projection of spin in the momentum direction, it is

$$h = \frac{\vec{s} \cdot \vec{p}}{|\vec{p}|} = \frac{\vec{\alpha} \cdot \vec{p}}{|\vec{p}|}, \quad (45)$$

and

$$\vec{\alpha} \cdot \vec{p} = \alpha_x p_x + \alpha_y p_y + \alpha_z p_z = \begin{pmatrix} 0 & -ip_z & ip_y \\ ip_z & 0 & -ip_x \\ -ip_y & ip_x & 0 \end{pmatrix}, \quad (46)$$

the $\vec{\alpha} \cdot \vec{p}$ eigenvalue problem $\vec{\alpha} \cdot \vec{p}\psi = \lambda\psi$ is

$$\begin{pmatrix} 0 & -ip_z & ip_y \\ ip_z & 0 & -ip_x \\ -ip_y & ip_x & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = \lambda \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (47)$$

therefore the characteristic equation is

$$\begin{vmatrix} -\lambda & -ip_z & ip_y \\ ip_z & -\lambda & -ip_x \\ -ip_y & ip_x & -\lambda \end{vmatrix} = 0, \quad (48)$$

i.e.,

$$\lambda(\vec{p}^2 - \lambda^2) = 0, \quad (49)$$

the roots λ are

$$\lambda = |\vec{p}|, \quad -|\vec{p}|, \quad (50)$$

and the helicity h are

$$\hbar = +1, \quad -1. \quad (51)$$

When $\lambda = +1$ the photon is called right-handed photon, and when $\lambda = -1$ the photon is called left-handed photon.

5 The probability conservation equation of photon

In the following, we should give the probability density and probability conservation equation of photon.

The hermitian conjugate of (4) is

$$-i\hbar \frac{\partial \psi^\dagger}{\partial t} = i\hbar c \vec{\nabla} \psi^\dagger \cdot \vec{\alpha}, \quad (52)$$

multiplying (52) by ψ , there is

$$-i\hbar \frac{\partial \psi^\dagger}{\partial t} \psi = i\hbar c \vec{\nabla} \psi^\dagger \cdot \vec{\alpha} \psi, \quad (53)$$

multiplying (4) by ψ^\dagger , there is

$$i\hbar \psi^\dagger \left(\frac{\partial \psi}{\partial t} \right) = -i\hbar c \psi^\dagger \vec{\alpha} \cdot \vec{\nabla} \psi, \quad (54)$$

taking the difference, we get

$$i\hbar \left(\psi^\dagger \frac{\partial \psi}{\partial t} + \frac{\partial \psi^\dagger}{\partial t} \psi \right) + i\hbar c \psi^\dagger \vec{\alpha} \cdot (\vec{\nabla} \psi) + i\hbar c \vec{\nabla} \psi^\dagger \cdot \vec{\alpha} \psi = 0, \quad (55)$$

or

$$\frac{1}{c} \frac{\partial}{\partial t} (\psi^\dagger \psi) + \psi^\dagger \vec{\alpha} \cdot (\vec{\nabla} \psi) + (\vec{\nabla} \psi^\dagger) \cdot \vec{\alpha} \psi = 0, \quad (56)$$

i.e.,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0, \quad (57)$$

where

$$\rho = \psi^\dagger \psi, \quad \vec{J} = c \psi^\dagger \vec{\alpha} \psi, \quad (58)$$

are the probability and probability current density of photon, respectively.

6 The plane wave of free photon

We have the spiron equation of free photon

$$i\hbar \frac{\partial}{\partial t} \psi = H \psi, \quad (59)$$

where

$$H = c \vec{\alpha} \cdot \vec{p}, \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad (60)$$

since $\frac{\partial H}{\partial t} = 0$ and $[\vec{p}, H] = 0$, the energy E and momentum \vec{p} of photon are conserved quantity. Their common eigenstate is

$$\psi_{E,\vec{p}}(\vec{r}, t) = u(\vec{p}) e^{i(\vec{p} \cdot \vec{r} - Et)/\hbar}, \quad (61)$$

where

$$u(\vec{p}) = \begin{pmatrix} u_1(\vec{p}) \\ u_2(\vec{p}) \\ u_3(\vec{p}) \end{pmatrix}, \quad (62)$$

substituting equations (61) and (62) into (59), we have

$$c \vec{\alpha} \cdot \vec{p} u(\vec{p}) = E u(\vec{p}), \quad (63)$$

i.e.,

$$(c\alpha_x p_x + c\alpha_y p_y + c\alpha_z p_z) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = E \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad (64)$$

and expanding (64), we get

$$\begin{pmatrix} 0 & -icp_z & icp_y \\ icp_z & 0 & -icp_x \\ -icp_y & icp_x & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = E \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad (65)$$

or

$$Eu_1 + icp_z u_2 - icp_y u_3 = 0, \quad (66)$$

$$icp_z u_1 - Eu_2 - icp_x u_3 = 0, \quad (67)$$

$$icp_y u_1 - icp_x u_2 + Eu_3 = 0, \quad (68)$$

therefore the characteristic equation is

$$\begin{vmatrix} E & icp_z & -icp_y \\ icp_z & -E & -icp_x \\ icp_y & -icp_x & E \end{vmatrix} = 0, \quad (69)$$

the roots E are

$$E_1 = +c|\vec{p}|, \quad E_2 = -c|\vec{p}|. \quad (70)$$

From equations (67) and (68), we have

$$icp_zp_yu_1 - Ep_yu_2 - icp_xp_yu_3 = 0, \quad (71)$$

$$icp_y p_z u_1 - icp_x p_z u_2 + Ep_z u_3 = 0, \quad (72)$$

taking the difference of equations (71) and (72), we get

$$(Ep_y - icp_x p_z)u_2 + (Ep_z + icp_x p_y)u_3 = 0, \quad (73)$$

i.e.,

$$\frac{u_2}{u_3} = -\frac{Ep_z + icp_x p_y}{Ep_y - icp_x p_z}, \quad (74)$$

substituting (74) into (66), there is

$$\frac{u_1}{u_3} = \frac{ic(p_y^2 + p_z^2)}{Ep_y - icp_x p_z}, \quad (75)$$

by (74) and (75), we have

$$\frac{u_1}{u_2} = -\frac{ic(p_y^2 + p_z^2)}{Ep_z + icp_x p_y}, \quad (76)$$

the $u(p)$ can be written as

$$u(p) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = N \begin{pmatrix} ic(p_y^2 + p_z^2) \\ -(Ep_z + icp_x p_y) \\ Ep_y - icp_x p_z \end{pmatrix}, \quad (77)$$

where N is normalization constant, it can be obtained by the normalization

$$\begin{aligned} u^\dagger(p)u(p) &= N^2 \begin{pmatrix} -ic(p_y^2 + p_z^2) & -(Ep_z - icp_x p_y) & (Ep_y + icp_x p_z) \end{pmatrix} \begin{pmatrix} ic(p_y^2 + p_z^2) \\ -(Ep_z + icp_x p_y) \\ Ep_y - icp_x p_z \end{pmatrix} \\ &= 2E^2 N^2 (p_y^2 + p_z^2) = 1, \end{aligned} \quad (78)$$

i.e.,

$$N = \sqrt{\frac{1}{2E^2(p_y^2 + p_z^2)}}, \quad (79)$$

then

$$u(p) = \sqrt{\frac{1}{2E^2(p_y^2 + p_z^2)}} \begin{pmatrix} ic(p_y^2 + p_z^2) \\ -(Ep_z + icp_x p_y) \\ Ep_y - icp_x p_z \end{pmatrix}, \quad (80)$$

The plane wave solution of free photon is

$$\psi_{E,\vec{P}}(\vec{r},t) = \sqrt{\frac{1}{2E^2(p_y^2 + p_z^2)}} \begin{pmatrix} ic(p_y^2 + p_z^2) \\ -(Ep_z + icp_x p_y) \\ Ep_y - icp_x p_z \end{pmatrix} e^{i(\vec{p}\cdot\vec{r} - Et)/\hbar}. \quad (81)$$

7 The spin wave function of photon

From equations (41) and (42), we can find \vec{s}^2 commute with s_x , s_y and s_z , we should calculate the common eigenstate of \vec{s}^2 and s_z , they are

$$\vec{s}^2 \chi_\mu = 2\hbar^2 \chi_\mu, \quad (82)$$

$$s_z \chi_\mu = \mu \hbar \chi_\mu, \quad (83)$$

where the common eigenstate $(\chi_\mu)^T = (\varphi_1, \varphi_2, \varphi_3)$, the equation (83) can be written as

$$\begin{pmatrix} 0 & -i\hbar & 0 \\ i\hbar & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \mu \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}, \quad (84)$$

therefore the characteristic equation is

$$\begin{vmatrix} -\mu & -i & 0 \\ i & -\mu & 0 \\ 0 & 0 & -\mu \end{vmatrix} = 0, \quad (85)$$

i.e.,

$$-\mu(\mu^2 - 1) = 0, \quad (86)$$

the roots μ are

$$\mu_1 = 0, \quad \mu_2 = 1, \quad \mu_3 = -1, \quad (87)$$

substituting $\mu_1 = 0$ into (84), we get

$$\begin{cases} -i\varphi_2 = 0 \\ i\varphi_1 = 0 \end{cases}, \quad (88)$$

i.e.,

$$\varphi_1 = \varphi_2 = 0, \quad \varphi_3 \neq 0, \quad (89)$$

the normalization spin wave function is

$$\chi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (90)$$

substituting $\mu_2 = 1$ into (84), we get

$$\begin{cases} -i\varphi_2 = \varphi_1 \\ i\varphi_1 = \varphi_2 \\ \varphi_3 = 0 \end{cases}, \quad (91)$$

we can take

$$\varphi_1 = -1, \quad \varphi_2 = -i, \quad \varphi_3 = 0, \quad (92)$$

the normalization spin wave function is

$$\chi_1 = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad (93)$$

substituting $\mu_3 = -1$ into (84), we get

$$\begin{cases} -i\varphi_2 = -\varphi_1 \\ i\varphi_1 = -\varphi_2 \\ \varphi_3 = 0 \end{cases}, \quad (94)$$

we can take

$$\varphi_1 = 1, \quad \varphi_2 = -i, \quad \varphi_3 = 0, \quad (95)$$

the normalization spin wave function is

$$\chi_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}, \quad (96)$$

and these spin wave functions satisfy the normalization condition

$$\sum_{\alpha} \chi_{\mu}^*(\alpha) \chi_{\mu'}(\alpha) = \delta_{\mu\mu'}. \quad (97)$$

8 The spiron wave equation of non-free photon

In view of the above, we have given the spiron wave equation of free photon. In the following, we should give the spiron wave equation of non-free photon.

For the non-free particle, the Einstein's dispersion relation is

$$(E - V)^2 = c^2 \vec{p}^2 + m_0^2 c^4, \quad (98)$$

factorizing (98), we obtain

$$(E - V)^2 - c^2 \vec{p}^2 - m_0^2 c^4 = (E - V - c\vec{p} \cdot \vec{\alpha} - m_0 c^2 \beta)(E - V + c\vec{p} \cdot \vec{\alpha} + m_0 c^2 \beta). \quad (99)$$

For photon, $m_0 = 0$, equation (99) becomes

$$(E - V)^2 - c^2 \vec{p}^2 = (E - V - c\vec{p} \cdot \vec{\alpha})(E - V + c\vec{p} \cdot \vec{\alpha}) = 0, \quad (100)$$

or

$$(E - V - c\vec{p} \cdot \vec{\alpha}) = 0, \quad (101)$$

canonical quantization equation (101), we have

$$i\hbar \frac{\partial}{\partial t} \psi = -i\hbar c \vec{\alpha} \cdot \vec{\nabla} \psi + V \psi, \quad (102)$$

the potential energy of photon in medium is [17]

$$V = \hbar\omega(1 - n), \quad (103)$$

the spiron equation of photon in medium is

$$i\hbar \frac{\partial}{\partial t} \psi = -i\hbar c \vec{\alpha} \cdot \vec{\nabla} \psi + \hbar\omega(1 - n)\psi, \quad (104)$$

by the method of separation variable

$$\psi(\vec{r}, t) = \psi(\vec{r})f(t), \quad (105)$$

the equation (104) becomes

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r})f(t) = [-i\hbar c \vec{\alpha} \cdot \vec{\nabla} + \hbar\omega(1 - n)]\psi(\vec{r})f(t), \quad (106)$$

or

$$\frac{i\hbar \frac{\partial}{\partial t} f(t)}{f(t)} \psi^\dagger(\vec{r})\psi(\vec{r}) = \psi^\dagger(\vec{r})[-i\hbar c \vec{\alpha} \cdot \vec{\nabla} + \hbar\omega(1 - n)]\psi(\vec{r}), \quad (107)$$

or

$$\frac{i\hbar\frac{\partial}{\partial t}f(t)}{f(t)} = \frac{\psi^\dagger(\vec{r})[-i\hbar\vec{\alpha} \cdot \vec{\nabla} + \hbar\omega(1-n)]\psi(\vec{r})}{\psi^\dagger(\vec{r})\psi(\vec{r})} = E, \quad (108)$$

we have

$$f(t) = f_0 e^{-iEt/\hbar}, \quad (109)$$

and

$$\psi^\dagger(\vec{r})[-i\hbar\vec{\alpha} \cdot \vec{\nabla} + \hbar\omega(1-n)]\psi(\vec{r}) = E\psi^\dagger(\vec{r})\psi(\vec{r}), \quad (110)$$

i.e.,

$$[-i\hbar\vec{\alpha} \cdot \vec{\nabla} + \hbar\omega(1-n)]\psi(\vec{r}) = E\psi(\vec{r}), \quad (111)$$

where E is photon total energy. Equation (111) is time-dependent spiron equation of photon, and the general spiron equation of photon in arbitrary potential energy V is

$$[-i\hbar\vec{\alpha} \cdot \vec{\nabla} + V]\psi(\vec{r}) = E\psi(\vec{r}). \quad (112)$$

9 Conclusion

In this paper, we have proposed the spiron equation of free and non-free photon, and give the spin operator and spin wave function of photon. We can further study two and multi-photon spin wave function, and can obtain multi-photon entanglement state. We calculate the helicity of photon and prove there are left-handed and right-handed photon. By the spiron equation of non-free photon, we can study the quantum property of photon in medium. We think the spiron equation of free and non-free photon can be widely used in the future.

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